CE 5890 Final Project

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Introduction

- 2-D Plume tracking in the designed water body by using Stommel's general circulation equation
- Add a fixed plume source (constant flow) the circulation, and using numerical method (Finite Difference) to solve the movement of plume

Stommel's General Circulation Equation

- Analytical solution

$$\psi = F\sin(\frac{\pi y}{b})(pe^{Ax} + qe^{-Bx} - 1)$$

- Eastward current (u)

$$u = \frac{\partial\varphi}{\partial y} = F\cos(\frac{\pi y}{b})(\frac{\pi}{b})(pe^{Ax} + qe^{-Bx} - 1)$$

- Northward current (v)

$$v = \frac{\partial\varphi}{\partial x} = -F\sin(\frac{\pi y}{b})(Ape^{Ax} + Bqe^{-Bx})$$

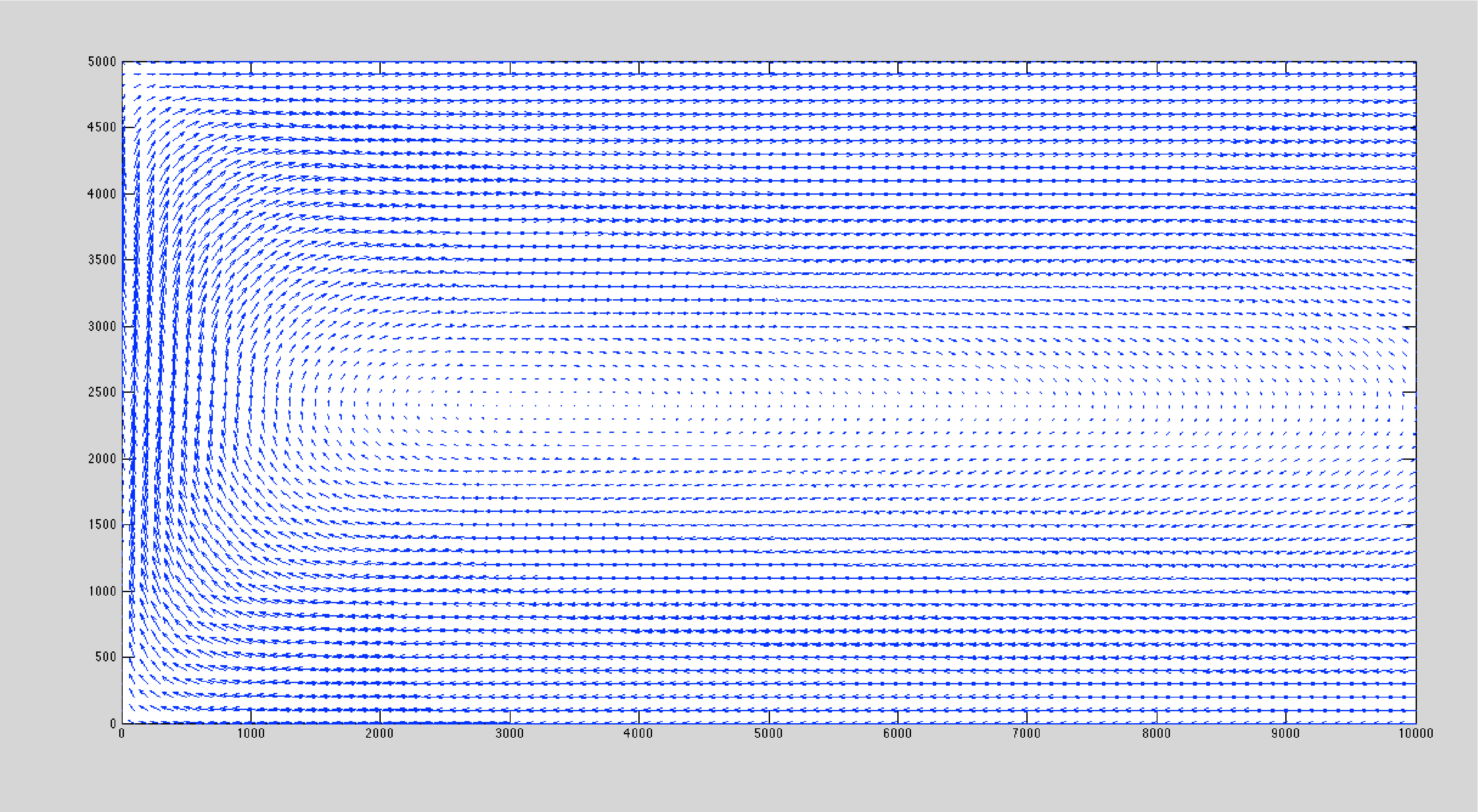
Setting parameters

- $F = 1$ km2/sec
- $p = 0.05$
- $q = 0.95$
- $A = 0.0002$ km^(-1)
- $B = 0.0013$ km^(-1)
- $X = 10000$ km
- $Y = b = 5000$ km

Numerical settings

- dx = 100 km
- dy = 100 km
- dt = 10800 sec
- nt = 2160
- Tracer at x=3000 km, y=1000 km
- k = 0.1,0.01,0.001 (sensitive analysis)

Flow field from analytical solution



The plume source (as tracer)

- Tracking PDE

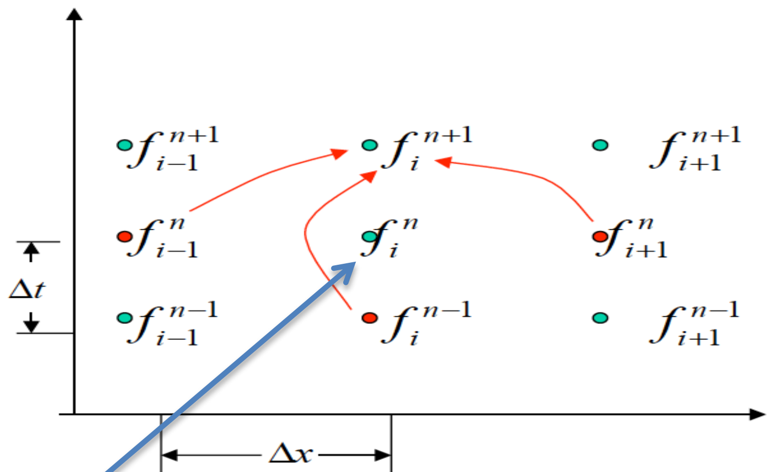
$$\varphi_t + u\varphi_x + v\varphi_y = k(\varphi_{xx} + \varphi_{yy}) + S(x, y)$$

- Source S(x,y)
  - 1 unit of tracer/hour
  - Designed at fixed location
    - x=3000 km, y=1000 km

Finite Difference Numerical Method

- Leap-frog scheme
  - Centered time
  - Centered space

$$f_i^{n+1} = f_i^{n-1} - C\frac{\Delta t}{\Delta x}(f_{i+1}^n - f_{i-1}^n)$$



- Forward time/centered space
  - Used to calculate 2nd time step

Leap-frog for Advection and Diffusion

- Advection

$$f_x^{t+1} = f_x^{t-1} - C\frac{\Delta t}{\Delta x}[f_{x+1}^t - f_{x-1}^t]$$

- Diffusion
  - Unstable $f_x^{t+1} = f_x^{t-1} + k\frac{2\Delta t}{\Delta x^2}[f_{x+1}^t - 2f_x^t + f_{x-1}^t]$
  - Stable $f_x^{t+1} = f_x^{t-1} + k\frac{2\Delta t}{\Delta x^2}[f_{x+1}^{t-1} - 2f_x^{t-1} + f_{x-1}^{t-1}]$

Numerical Expression

- Leap-frog scheme

$$f_{x,y}^{t+1} = f_{x,y}^{t-1} + 2\Delta t * \{\frac{k}{\Delta x^2}[f_{x+1,y}^{t-1} - 2f_{x,y}^{t-1} + f_{x-1,y}^{t-1}] + \frac{k}{\Delta y^2}[f_{x,y+1}^{t-1} - 2f_{x,y}^{t-1} + f_{x,y-1}^{t-1}]$$

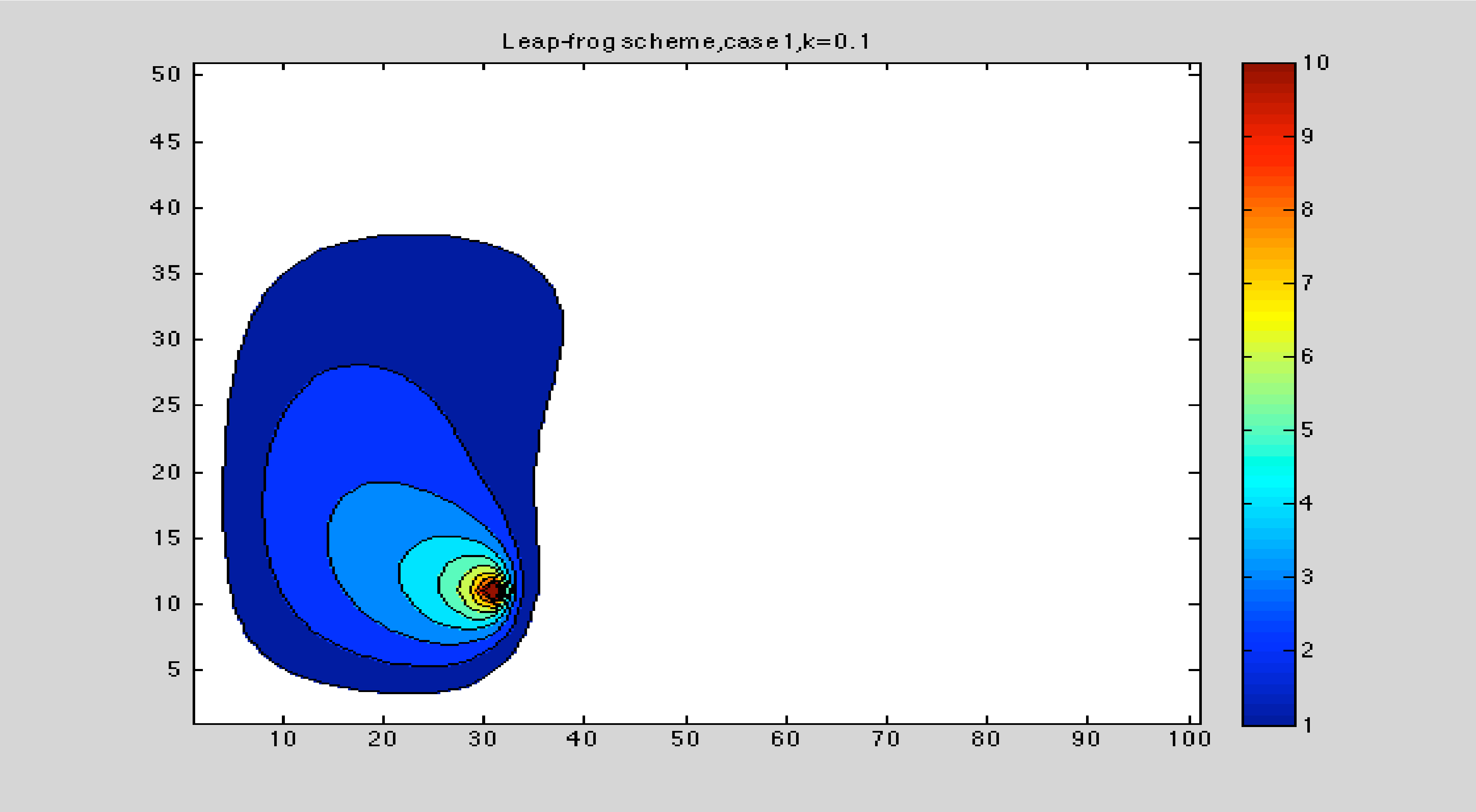
$$\ldots\ldots\ldots\ldots\ldots\ldots - \frac{u}{2\Delta x}[f_{x+1,y}^{t} - f_{x-1,y}^{t}] - \frac{v}{2\Delta x}[f_{x,y+1}^{t} - f_{x,y-1}^{t}]\} + Source * 2\Delta t$$

- Forward time/centered space
  - Used to calculate 2nd time step

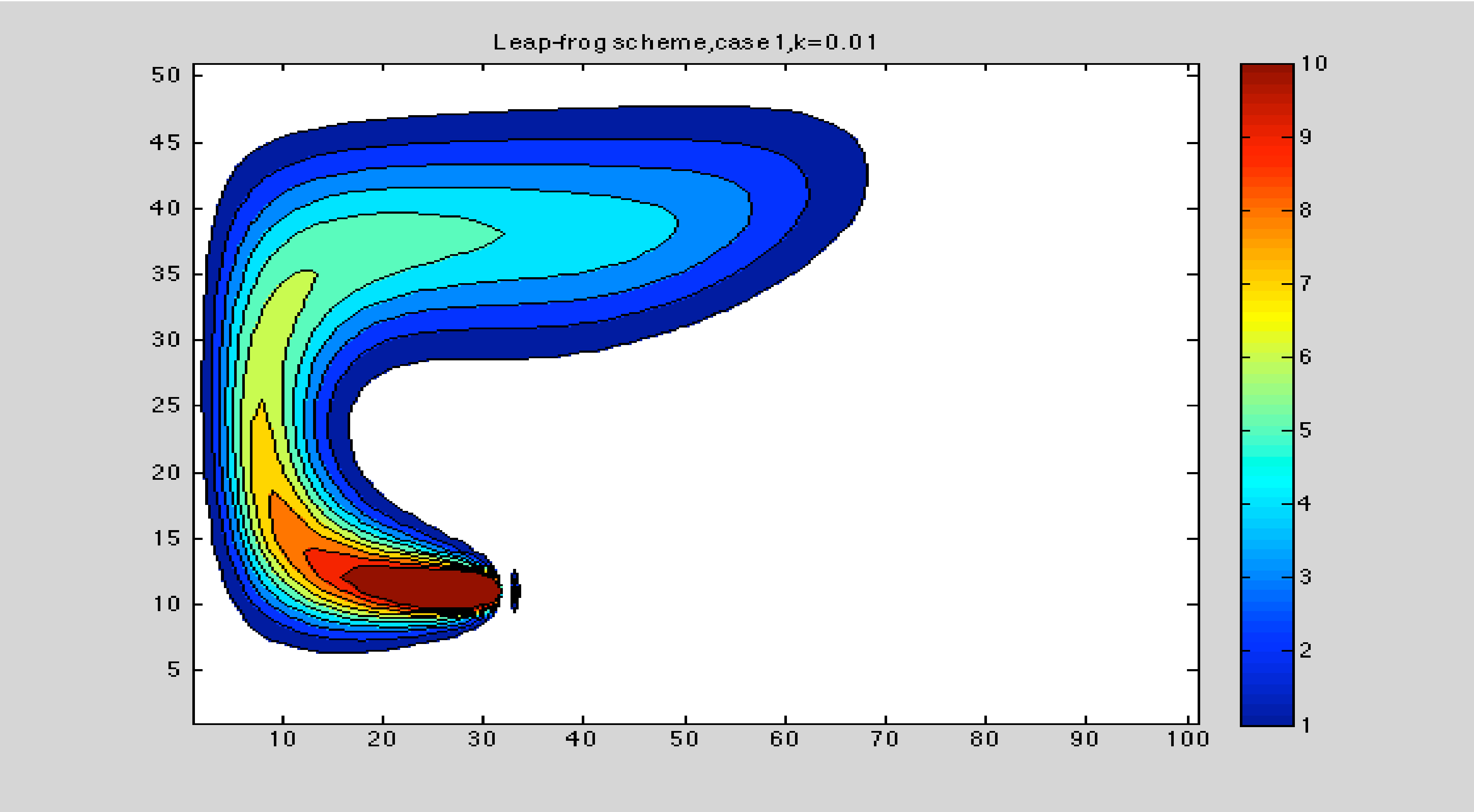
$$f_{x,y}^{t+1} = f_{x,y}^{t} + \Delta t * \{\frac{k}{\Delta x^2}[f_{x+1,y}^{t} - 2f_{x,y}^{t} + f_{x-1,y}^{t}] + \frac{k}{\Delta y^2}[f_{x,y+1}^{t} - 2f_{x,y}^{t} + f_{x,y-1}^{t}]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots - \frac{u}{2\Delta x}[f_{x+1,y}^{t} - f_{x-1,y}^{t}] - \frac{v}{2\Delta x}[f_{x,y+1}^{t} - f_{x,y-1}^{t}]\} + Source * 2\Delta t$$

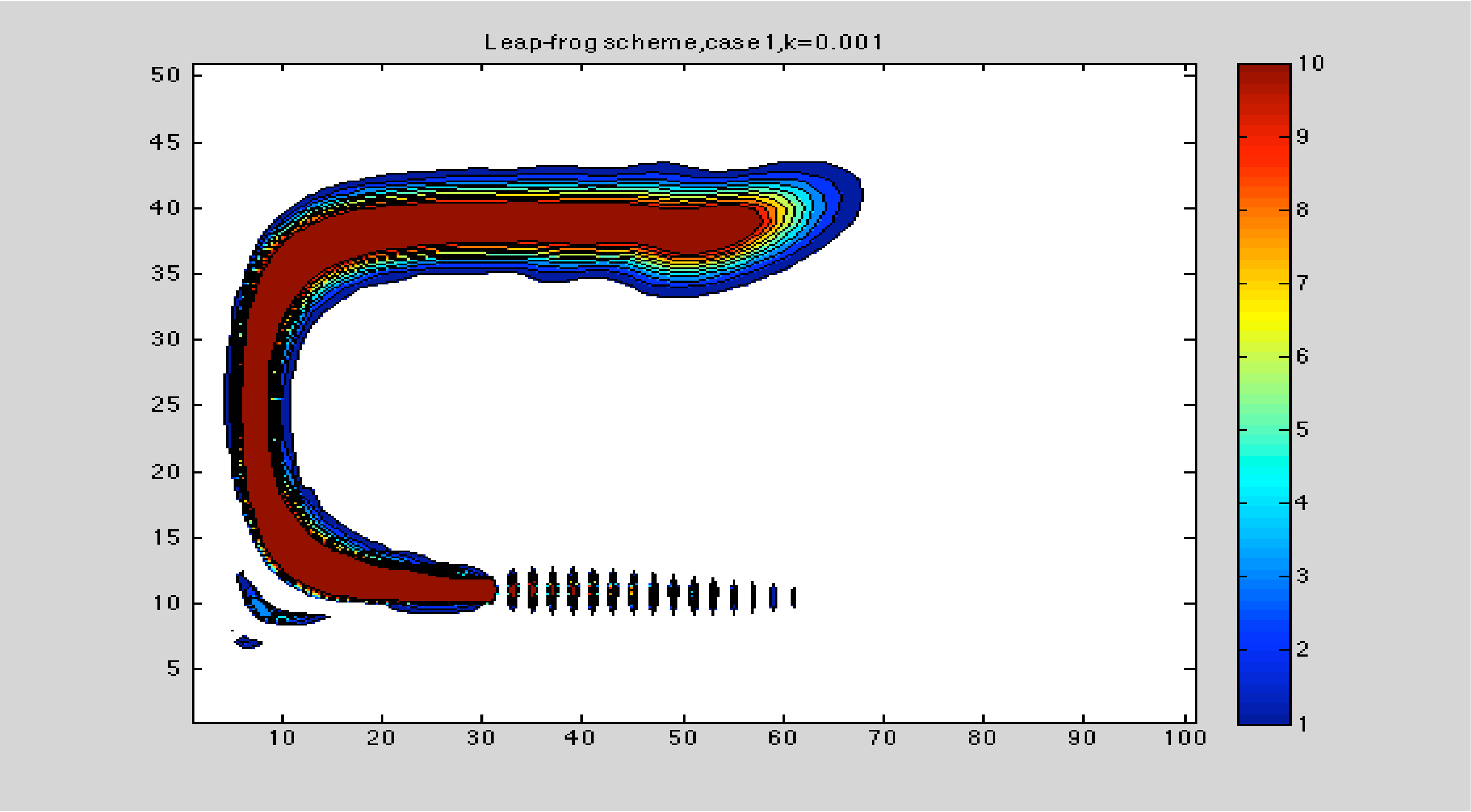
Modeling result_case: k=0.1



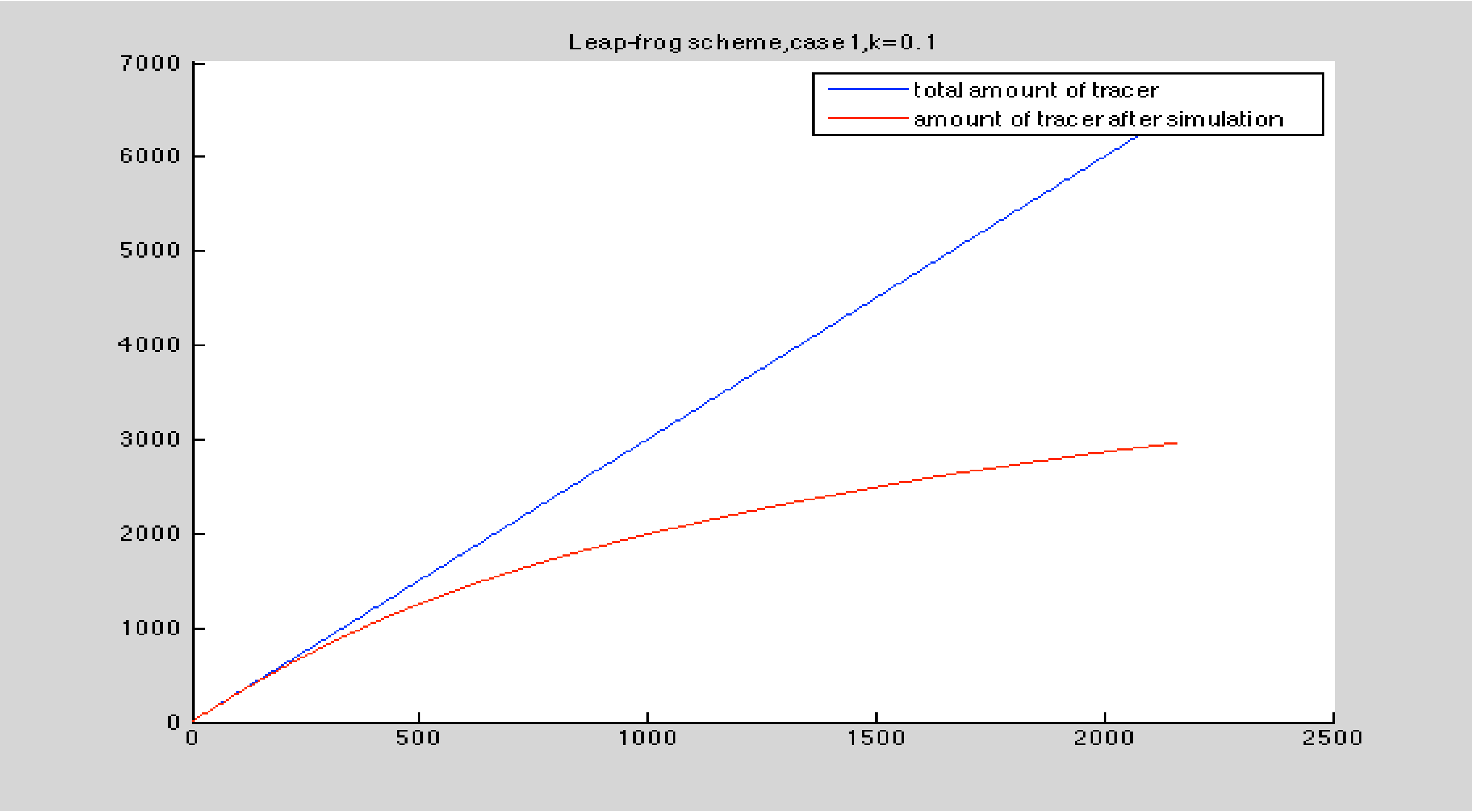
Modeling result_case: k=0.01



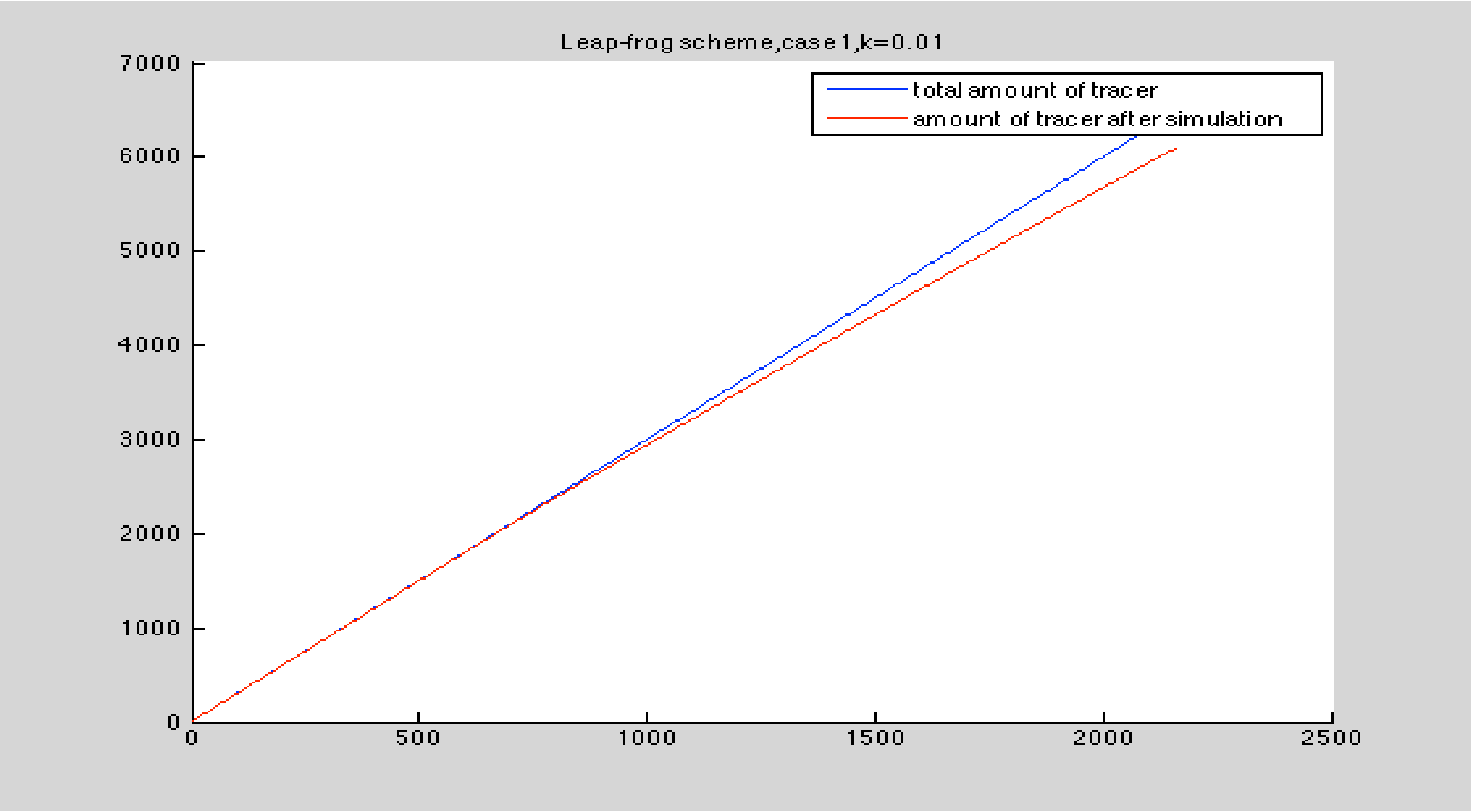
Modeling result_case: k=0.001



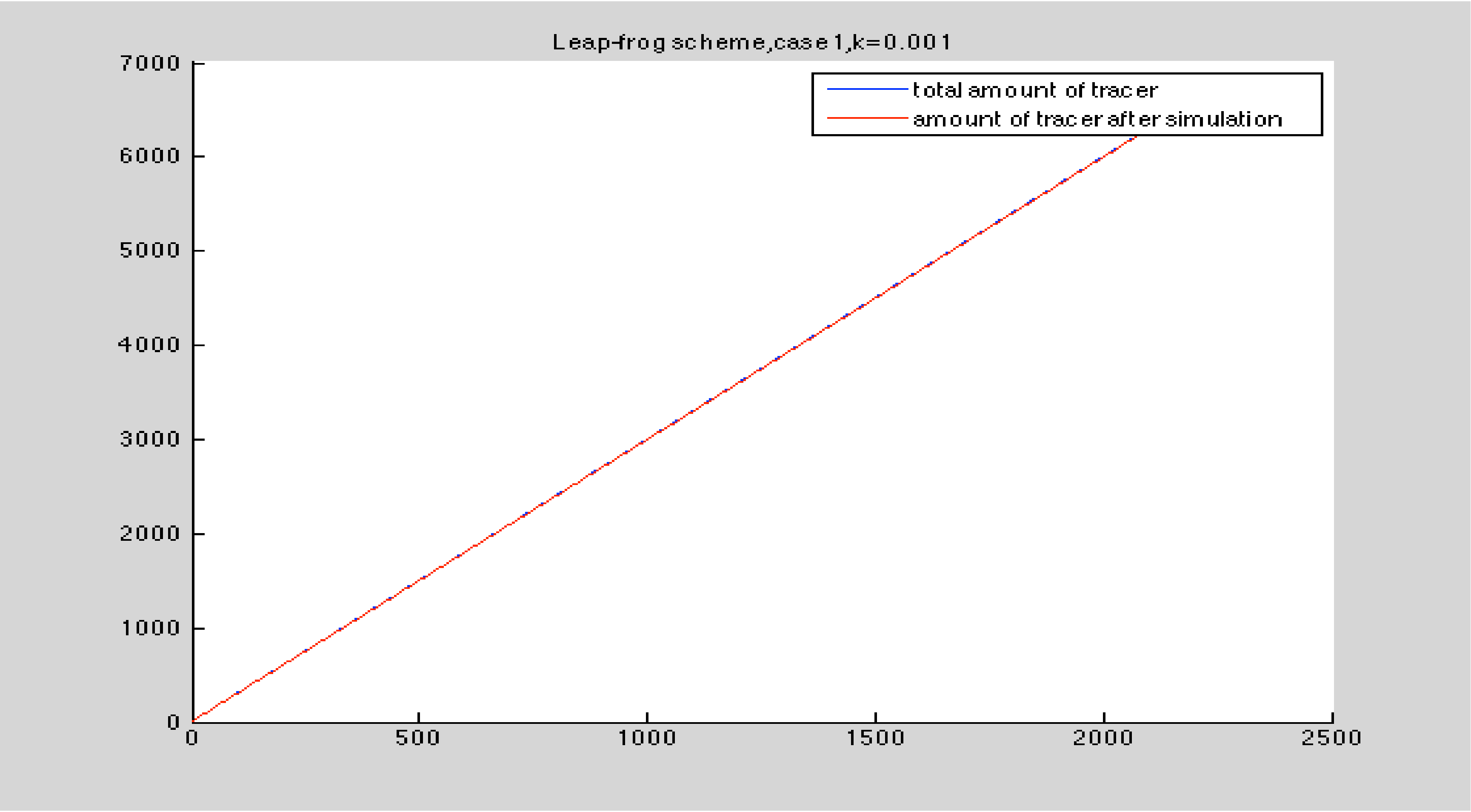
Modeling result, k=0.1



Modeling result, k=0.01



Modeling result, k=0.001



Discussion

- This is just a very simple model for 2-D plume tracking. The Source term is set as a constant value which released at each time step.
- No temperature/heat term was added

- Diffusion coefficient analysis
  - As the diffusive coefficient is increasing
    - the path of tracer is getting more widely
    - the oscillation is decreasing
    - the numerical damping is increasing

Conclusion

- This model is able to track plume pattern by comparing the model output to the analytical current field
- Finite Difference Method (Leap-frog scheme)
  - Working for this 2-D model
  - Conditional stable for advection equation
  - Has a ‘trap’ when applying to diffusive term, in normal format, it is unconditional unstable for diffusive equation, and needs to be modified for stability (use ‘t-1’ instead of ‘t’)
  - Has numerical oscillation
- Diffusive coefficient is sensitive for the plume path, and is proportional to the numerical damping, and has opposite ratio to the numerical oscillation

Further Development and Application

- The idealized 2-D plume tracking program will need to have further development when applying to the real world situation.
  - Add vertical dimension
  - Add time and space vary forcing condition
    - Measured data (e.g., wind, temperature)
  - Multi-flow-equation may be needed for specific situation

Thank you!

- Question?